A CRASH COURSE IN GENERAL RELATIVITY AND BLACK HOLES

OLUMAKINDE OGUNNAIKE

1. Special Relativity and 4-vectors

In this section, we'll blitz through an introduction to special relativity. We'll focus on the odd properties and geometric relations of the relativistic *spacetime* that replaces old notions of distinct space and time. Along the way, it will be useful to introduce some new notation that will become indispensable in later sections.

1.1. Relativity and Reference Frames.

Much of physics is about finding constants. The world is a complicated place, so, to make modeling it easier, we look for things that are less complicated. Things that are symmetric under rotations or constant over time are some of the best examples of this. In these cases, it shouldn't matter if we look at our system at an angle or after an hour, respectively. We might still describe events with angles and times, but that's more a matter of convention than necessity. But, for better or worse, we perceive the world in a conventional way (we see distances from ourselves, and imagine times measured from the present). We just hope that some aspects of how we perceive the world (like the fundamental physical laws that determine physics) aren't *too* dependent on the person describing them. The laws of physics should be objective things.

To make this all a bit more concrete, we'll introduce the concept of **reference frames**. Roughly speaking these are the descriptions of the world from the perspective of a person (aka an observer), who considers themself to be at rest. So if I am in a boat traveling towards my friend on the shore, in my reference frame, he *and* the shore are both actually moving towards me! Essentially, to translate between two observers reference frames is simply to translate between different conventions of perceiving or describing the world. A natural question you might ask is this: *when is one convention better than another?* After all, if I see both my friend and the shoreline coming closer to me, I should reasonably deduce that *I* am actually the one moving. This might seem reasonable due to experience, but according to physics, there's nothing to say the whole shoreline wouldn't be moving. The real reason you could likely tell you were moving would be the ripples in the water behind you or the rocking of the boat. If these were not there, you would hope that the laws of physics would not care if you were moving or still.

The above idea is called the **principle of relativity**. It says that there is no "preferred" inertial (non-accelerating) reference frame. So as long as I move at a constant velocity, the physics I see should be the same physics you see.[1] But that doesn't change the fact that the distances and times in one reference frame will be different from those in another. We need a way to consistently translate between them. The most familiar and intuitive transformation between reference frame is that of **Galilean Transformations**. The reference frame of someone moving forward at speed v in the x-direction is simply

$$\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned} \tag{1.1.1}$$

These are really easy to understand. Essentially, they just say that if I want to describe things in the reference frame of someone moving at a velocity, v, in front of me, I can just use my own description and subtract off the rate at which this reference frame is moving forward, vt, from any positions I measure. What's important to notice in this transformation is that we held *time* constant between reference frames. This is a great approximation of our daily experience, but in reality, isn't quite right....

1.2. **Lorentz Transformation.** Back in the 1800's and around the turn of the century, physicists like Maxwell and Einstein figured out that the equations governing Electromagnetism told a funny story. They said that the speed at which light travels was constant, regardless of reference frame. That speed, though, is so fast, it's not surprising we missed it for so long:

$$c = 299,792,458m/s \tag{1.2.1}$$

It doesn't take much work to see how this messes up the Galilean transformations above. But it will take a little work to show how we can construct transformations that respect a constant speed of light instead of a constant rate of the passage of time.

To get a feel for this, we can look at what happens to time in reference frames moving with respect to each other. If we consider a "light clock," where light is bouncing between two mirrors, we notice that it will look very different if we are moving with respect to the mirrors. In frame (a), it travels up an down a distance, h, while in frame (b) it also travels across a distance vt.[2]

In frame (a), the light returns to its original position after a time $\Delta t = \frac{2h}{c}$. But in frame (b), this time is a little more complicated:

[1]For an example of a non-inertial reference frame, think of spinning around on a merry-go-round. As long as you stay at a fixed radius, you'll feel a centrifugal force pushing outwards, like an extra gravitational force. But someone outside the merry-go-round, this extra, ever-present force isn't present, so according to special relativity, you experience different physics.

[2]It is important that the time "t" is measured in the moving frame where the mirrors are stationary. Can you see why?

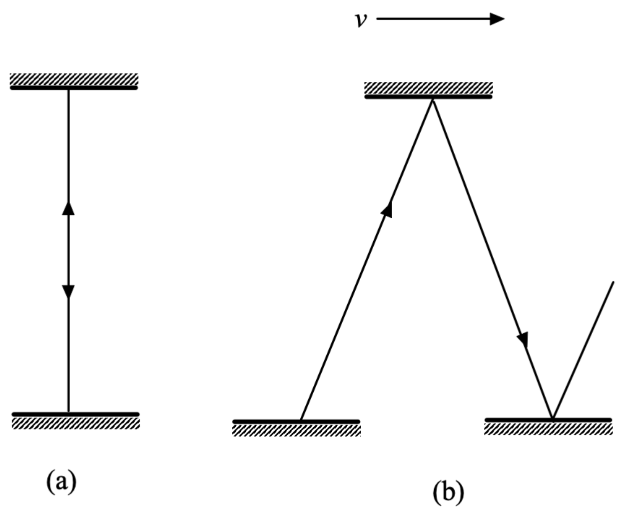

FIGURE 1.1. A light clock in a relatively stationary reference frame, and one moving ar a relative velocity "v"

$$\Delta t' = \frac{2}{c}\sqrt{h^2 + (v\Delta t')^2}, \quad \Rightarrow \quad \Delta t' = \frac{2h/c}{\sqrt{1-\frac{v^2}{c^2}}} \tag{1.2.2}$$

But if you remember what the time for this period was in the original frame ($\Delta t = \frac{2h}{c}$), this can be re-written to give us the famous time-dilation formula:

$$\Delta t' = \gamma \Delta t = \frac{\Delta t}{\sqrt{1-\frac{v^2}{c^2}}} \tag{1.2.3}$$

Here we introduce the *lorentz factor*: $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$. It appears so often in relativity that it gets its own symbol. Notice that $\gamma \geq 1$, so that time intervals in a moving frame are always *bigger* than they are in a stationary one. From this, it is easy to figure out how space acts. If we look at a point at a distance Δx away from us in a stationary frame, then in a frame moving at velocity v towards the point (so that it reaches us in time Δt), it will actually be at a distance:

$$\Delta x' = v\Delta t = v\Delta t'\sqrt{1-\frac{v^2}{c^2}} = \frac{\Delta x}{\gamma} \tag{1.2.4}$$

And we have the famous length contraction formula. It turns out these can be made more general by introducing **Lorentz Transformations**. These take a bit more time to derive properly, but let's start with the most general type of linear transformation and see if we can't figure them out.[3]:

$$x' = ax + bt$$
$$t' = dx + et \tag{1.2.5}$$

We will call the reference frame with unprimed coordinates S, and that with primed coordinates S'. Now, if the origin in S (x = 0) moves at speed -v in the S' frame, then the transformation above tells us two things:

$$x' = bt, \quad t' = et \quad \Rightarrow \quad \frac{dx'}{dt'} = \frac{b}{e} = -v \tag{1.2.6}$$

But since $v = \frac{dx}{dt} = \frac{b}{a}$, we can see that a = e, and $b = -av$:

$$x' = ax - avt$$
$$t' = dx + at \tag{1.2.7}$$

Now, if we require light to move at the same speed in both reference frames, then when $x' = ct'$, $x = ct$ so

$$ct' = a(ct) - avt$$
$$t' = dct + at \tag{1.2.8}$$

Which allows us to solve for $d = \frac{-av}{c^2}$. One last thing is needed to give us our lorentz transformations. To solve for a, we can check that the transformation works in reverse, solving for x and t in terms of x' and t', or simply guess the right answer from the contraction formulas found above:

$$x' = \gamma(x - vt)$$
$$t' = \gamma(-\frac{vx}{c^2} + t) \tag{1.2.9}$$

Framed in terms of matricies, this becomes:

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -v \\ -\frac{v}{c^2} & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \tag{1.2.10}$$

[3]Why is it essential that the transformation is linear? It has to do with inertial reference frames moving at constant speeds....

1.3. **The spacetime interval.** Now that we've gone over transformations, we can begin to talk about spacetime geometry. Geometry doesn't care about coordinates, only lengths and angles. A square rotated 45° to look like a diamond still has the same geometry, even if we change perspective. This means that if we rotate a coordinate system (x,y) by some angle θ, it shouldn't change the lengths or angles in the square. We can write a rotation of coordinates as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{1.3.1}$$

Notice that this is very similar to the Lorentz transformation defined above. But in our 2D planar geometry, as we said, there was something that didn't depend on the rotations above: length. And if we recall the (square of the) length in 2D can be defined as:

$$(\Delta r)^2 = (\Delta x)^2 + (\Delta y)^2 \tag{1.3.2}$$

As an excercise, we put in rotated coordinates to check that the length really doesn't change[4]:

(1.3.3)

$$\begin{aligned}
(\Delta r')^2 &= (\Delta x')^2 + (\Delta y')^2 \\
&= (\Delta x\cos\theta - \Delta y\sin\theta)^2 + (\Delta y'\cos\theta + \Delta x\sin\theta)^2 \\
&= (\Delta x)^2(\cos^2\theta + \sin^2\theta) + (\Delta y)^2(\cos^2\theta + \sin^2\theta) + (\Delta x)(\Delta y)(-\cos\theta\sin\theta + \cos\theta\sin\theta) \\
&= (\Delta x)^2 + (\Delta y)^2 = (\Delta r)^2
\end{aligned}$$

Now, we want to find something similar for the two spacetime variables x and t, but there's something a bit funny. The lorentz transformation gets even more odd, if we go through a little algebra, we can rewrite the lorentz transformation into something even more similar to the rotations above:

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} \cosh\eta & -\sinh\eta \\ -\sinh\eta & \cosh\eta \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \tag{1.3.4}$$

Where $\gamma = \cosh\eta$ and $\gamma v = \sinh\eta$. Now this looks like a rotation, but in hyperbolic coordinates. For these to work, we need a new definition of length. We define the **spacetime interval** as:

$$(\Delta s)^2 = (\Delta x)^2 - (c\Delta t)^2 \tag{1.3.5}$$

[4]This, and the invariance in the next section have one more requirement involving something called a determinant. To get a sense of how it works, try multiplying each matrix entry by a different variable, and check how they need to be related to give the original result.

Why the negative sign? Well it has something to do with what we define this to be a distance from. Before getting into that, though, lets check it is, indeed invariant under the lorentz transformations:

$$
\begin{aligned}
(\Delta s')^2 &= (\Delta x')^2 - (c\Delta t')^2 \\
&= \gamma^2(\Delta x - v\Delta t)^2 - c^2\gamma^2(\Delta t - \frac{\Delta x v}{c^2})^2 \\
&= \gamma^2\left((\Delta x)^2(1 - \frac{v^2}{c^2}) - (c\Delta t)^2(1 - \frac{v^2}{c^2})\right) \\
&= \gamma^2(1 - \frac{v^2}{c^2})(\Delta x)^2 - (c\Delta t)^2 \\
&= (\Delta x)^2 - (c\Delta t)^2 = (\Delta s)^2
\end{aligned}
\tag{1.3.6}
$$

Well this is certainly invariant, but it has the odd property that any point that satisfies $x = ct$ has a spacetime interval of $\Delta s = 0$. What this means is that the spacetime interval does not define a length in the normal sense, but in the same way that the length of a straight line measures the shortest possible distance between two points (say, the origin and another point (x,y)), the spacetime interval measures the shortest possible distance between two points (essentially by going to the frame where the two events happen at the same time). This takes some getting used to. A good way to try to do so is to think about the different ways coordinate systems change when they are rotated vs when they are boosted (giving a lorentz transform)

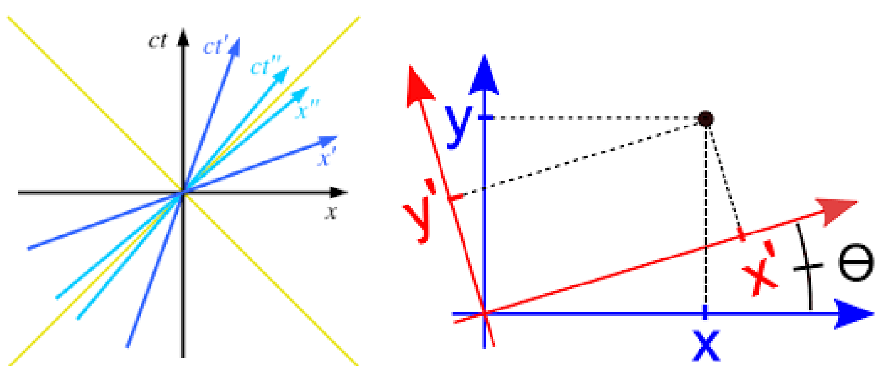

FIGURE 1.2. A minkowski Diagram for three different coordinate systems under boosts, and one for a rotated coordinate system

1.4. **4-vectors, metrics, and notation.**

This can be written in a slightly more compact way. We can re-write these two lengths by using a matrix to define an inner product:

$$
\begin{aligned}
(\Delta r)^2 &= \begin{pmatrix}\Delta x & \Delta y\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \sum_{ij}\delta_{ij}\Delta r^i \Delta r^j = (\Delta x)^2 + (\Delta y)^2 \\
(\Delta s)^2 &= \begin{pmatrix}c\Delta t & \Delta x\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}c\Delta t\\ \Delta x\end{pmatrix} = \sum_{\mu\nu}\eta_{\mu\nu}\Delta x^\mu \Delta x^\nu = -c^2(\Delta t)^2 + (\Delta x)^2
\end{aligned} \tag{1.4.1}
$$

Where we defined r^i such that $(r^1, r^2) = (x, y)$, and x^μ such that $(x^0, x^1) = (ct, x)$. Now we can define inner products for our two different geometries with δ_{ij} and $\eta_{\mu\nu}$. These are the **metric tensors** for the different geometries[5]. These define notions of distance in different spaces. Another familiar example might be the metric for $\mathbb{R}^3$ in spherical coordinates and that of the surface of a sphere S^2:

$$
\delta_{ij} = \begin{pmatrix}1 & 0 & 0\\ 0 & r^2 & 0\\ 0 & 0 & r^2\sin^2\theta\end{pmatrix}, \quad h_{ij} = \begin{pmatrix}1 & 0\\ 0 & \sin^2\theta\end{pmatrix} \tag{1.4.2}
$$

At this point, some comments on notation and conventions.

(1) At this point, since space and time are twisting together, it helps to define points by lumping together their 3 position coordinates and 1 time component. This way, an "event" in spacetime can be given a **4-vector** $x^\mu = (ct, x, y, z)^\mu$.
(2) since space and time are now on nearly the same footing, we notice that the speed of light is essentially just a unit conversion factor (like 2.54 in/cm). So from now on, we will just assume we are measuring everything in the appropriate units (you can think of this like measuring time in "light seconds"), or to put it another way, we will be setting c = 1 from now on.
(3) Finally, we will be doing lots of sums in future, especially with different metrics. Now that we have four components to keep track of, lots of things will be loaded into explicit sums. But this will still look messy. To avoid cluttering calculations with summation symbols, we will sum over any repeated indicies. This means that the length formula above can be written[6] $(\Delta\vec{r})^2 = \delta_{ij}\Delta r^i \Delta r^j = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$

Now metrics are extremely important to general relativity, so we might as well take some time to get comfortable with them. As the name implies, metrics are integrally linked to notions of measuring distance. But, as we've discussed before, distance doesn't depend on the coordinates we use. So although we've written out multiple examples of metrics in particular coordinates, These coordinates are just an arbitrary choice. To get a

[5]a tensor is simply something with a number of indicies that transforms linearly. Familiar examples of these are the 1-tensors known as vectors, and 2-tensors, known as matricies, like those we used to define metrics above.

[6]The convention is that a lower index counts the elements along a row, while an upper index counts the elements down a column. So a vector r^i is a column vector, a vector r_j is a row vector, and matricies should really be written as M_i^j. To raise or lower a matrix, you simply multiply by the corresponding metric. So that a row vector can be expressed as a column vector times the metric: $r_i = \delta_{ij} r^j$

better feel for this, lets think of three different ways of parametrizing the plane $\mathbb{R}^2$. First, we'll use cartesian coordinates (x,y), then polar coordinates (r, θ), and finally, a new set of coordinates (u, v) such that $u = x$ and $v = (x + y)/\sqrt{2}$. These last coordinates are just what would happen if we decided to bend the y-axis outwards at a 45° angle. Well, let's get the metric for $\mathbb{R}^2$ (given by δ_{ij}) in these coordinates. For cartesian coordinates, we need only look at Eq. (1.4.1):

$$(dr)^2 = (dx)^2 + (dy)^2 = \begin{pmatrix} dx & dy \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}, \quad \Rightarrow \delta_{ij}(x, y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.4.3}$$

To get the metric in other coordinates, we can simply substitute the definitions of x an y into the cartesian definition of length:

(1.4.4)

$$(dr)^2 = (d(r\cos\theta))^2 + (d(r\sin\theta))^2 = (dr\cos\theta - rd\theta\sin\theta)^2 + (dr\sin\theta + rd\theta\cos\theta)^2$$

$$= (\cos^2\theta + \sin^2\theta)(dr)^2 + (\cos^2\theta + \sin^2\theta)r^2(d\theta)^2, \quad \Rightarrow \boxed{\delta_{ij}(r, \theta) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}}$$

$$(dr)^2 = (du)^2 + (d(\sqrt{2}v - u))^2 = 2(du)^2 + 2(dv)^2 - 2\sqrt{2}dudv, \quad \Rightarrow \boxed{\delta_{ij}(u, v) = \begin{pmatrix} 2 & -\sqrt{2} \\ -\sqrt{2} & 2 \end{pmatrix}}$$

These are three different ways of describing the same 2D planar geometry. They all give the same length formula (this should be clear since we derived them all from the same length formula), and they also give the same notion of angles. This is clear for polar and cartesian coordinates, whose basis vectors (x,y,r, and θ) are orthogonal to the paired variable in their coordinate system. However, this is not the case for the (u,v) coordinate system. $\vec{u}$ and $\vec{v}$ are certainly not orthogonal, but we can find the angle between them using the inner product defined by the metric in their coordinate system:

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} = \left(\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & -\sqrt{2} \\ -\sqrt{2} & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) / (\sqrt{2} * \sqrt{2}) = -\frac{1}{\sqrt{2}} \tag{1.4.5}$$

From this, we can infer that the basis vectors are rotated by an angle of 135° with respect ot one another, just as we said previously from their description. This teaches us that just because two metrics have different entries (even if one has diagonal entries and one does not), it does not imply that they describe different notions of lengths and angles (different geometries). In fact, so long as you can write a well-defined transformation from one coordinate system to another over the entire surface, as we did here, the descriptions should be equivalent. However, if there is no nice transformation that makes the metric look like it does for the cartesian description $\delta_{ij}(x, y)$ for an entire surface[7], this is a sign

[7]Another way of saying this would be that when you put a matrix in diagonal form, the entries, called *eigenvalues*, shouldn't depend on any variables if your geometry is flat.

that the geometry described is something different. An example is the metric for the surface of a sphere:

$$h_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \tag{1.4.6}$$

It only looks like the cartesian metric at the equator ($\theta = \frac{\pi}{2}$). And this is a sign that it's not flat. We'll see later how we can explicitly quantify aspects of the geometry from the metric, like curvature. But first, let's start thinking about what it means to move in a spacetime whose "flat" metric has this odd negative entry for time

$$\eta(t, \vec{r})_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{\mu\nu}$$

2. Differential Geometry and geodesics

Now that we have a definition of length using the metric $\eta_{\mu\nu}$, we can start using calculus to figure out what these geometric entities mean. First, we're going to define **rest mass**, m, and **proper time**, τ. These are just the mass of an object and the time between events in the object's reference frame ($\Delta x = 0$). Proper time can be made more quantitative by defining $d\tau = ids$ so that the relative sign between space and time flip:

$$d\tau = \sqrt{dt^2 - d\vec{r}^2} = dt\sqrt{1 - v^2} = \frac{dt}{\gamma} \leq dt \tag{2.0.1}$$

Thus, since it is always less than or equal to any measured interval of time between two events, we see that the proper time is the minimum time between events.

2.1. 4-velocity.

Now, we've already seen how time and space aren't invariant under lorentz transformations. And it's easy to check that velocities $v = \frac{dx}{dt}$ aren't either. So now we want to find a better way of measuring rates of time. Enter **4-velocities**:

$$u^\mu = \frac{dx^\mu}{d\tau} = (\frac{dt}{d\tau}, \frac{d\vec{r}}{d\tau})^\mu = \frac{dt}{d\tau}(1, \vec{v})^\mu = \gamma(1, \vec{v})^\mu \tag{2.1.1}$$

Notice something funny about the four velocity. Because we defined proper time with that odd imaginary factor, $u^2 = \eta_{\mu\nu}u^\mu u^\nu = -1$. This is worth checking for yourself. Another way of saying this is that[8]:

$$d\tau^2 = -\eta_{\mu\nu}dx^\mu du^\nu \tag{2.1.2}$$

[8]From here on, we will use the "technically incorrect" notation $d\tau^2 = (d\tau)^2$

2.2. **4-momentum.** An easy extension of this is something called **4-momentum**:

$$p^\mu = mu^\mu = \gamma(m, m\vec{v})^\mu = (E, \vec{p})^\mu \tag{2.2.1}$$

This gives us a relativistic definition of momentum and energy, where the energy comes from the fact that we have set c = 1, so that at rest, the energy should be $E = mc^2$ (Thanks, Einstein!). But just like time and space, these will be different in moving reference frames in just the same way time and space will be different in different reference frames. This also gives us the famous relativistic energy-momentum formula:

$$p^2 = -(p^0)^2 + (\vec{p})^2 = -E^2 + p^2 = -m^2, \quad E = \sqrt{(\vec{p})^2 + m^2} \tag{2.2.2}$$

With these definitions, it should be relatively clear (pun slightly intended) how to do obtain relativistic definitions of all of our favorite quantities. Simply define the four vector and take derivatives with respect to proper time or the four-vector.

2.3. **Generalizing coordinates.** Up to this point, we have been using "nice" coordinates. Someone moving along through spacetime will follow a trajectory through spacetime given by $x^\mu(\tau)$, where τ is the proper time along their trajectory (or the time measured in their reference frame)[9], and a 4-velocity (tangent to their worldline) given by $u^\mu(\tau) = \frac{dx^\mu(\tau)}{d\tau}$. But we could just as easily have described the trajectory in different coordinates: $y^\mu(\tau)$. But to use special relativity, these cannot be just any new coordinates. Remember that the framework we developed thusfar only works if we are moving with respect to another observer at a constant velocity. Framed in the new relativistic language we've developed:

$$\frac{du^\mu}{d\tau} = \frac{d^2x^\mu}{d\tau^2} = 0 \tag{2.3.1}$$

Let's think about what this means in terms of coordinate transformations. If we want to transform from the x to the y coordinates, then, using the chain rule, the 4-velocity should change from $u^\mu = \frac{dx^\mu}{d\tau}$ to $v^\mu = \frac{dy^\mu}{d\tau} = \frac{dx^\nu}{d\tau}\frac{dy^\mu}{x^\nu} = u^\nu \frac{dy^\mu}{dx^\nu}$. If we plug this into the equation above, we get:

$$0 = \frac{dv^\mu}{d\tau} = \frac{du^\nu}{d\tau}\frac{dy^\mu}{dx^\nu} + u^\nu \frac{d}{d\tau}\left(\frac{dy^\mu}{dx^\nu}\right) = \frac{d}{d\tau}\left(\frac{dy^\mu}{dx^\nu}\right) \tag{2.3.2}$$

This essentially means that the Jacobian $\left(\frac{dy^\mu}{dx^\nu}\right)$ is constant over the worldline, or, to put it in more familiar terms, converting between reference frames only comes from constant shifts in the ratios of the definitions of time and space (i.e. different velocities). This means all we've done is transform our metric $\eta_{\mu\nu}$ by lorentz transforms (imagine rotating a metric for cartesian coordinates of $\mathbb{R}^2$). Well that's nice to know, but we want to be able to describe physics in coordinates where things are doing more interesting things. So in the spirit of what's been done, we now allow for more general coordinates. We could, for

[9] Why are these equivalent?

example, write down coordinates that are uniformly accelerating relative to another frame, or one rotating with respect to it.

The examples above don't work with special relativity, but it would be a shame to give up all that we've done till now, so we're going to stipulate something: if we look really closely, and only worry about things locally, we expect things to look like they did in special relativity. This idea is called the **equivalence principle**, and it is worth dwelling on for a bit.

Essentially, the equivalence principle says that whatever coordinates we use, however funny our perspective might be, if we look close enough, physics should look the same: spacetime should look "flat" like it would in special relativity. To make this quantitative, let's look at the geometry of a sphere:

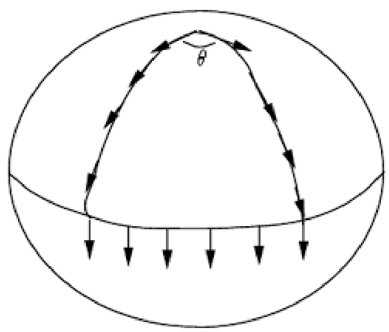

FIGURE 2.1. Vectors that are orthogonal at the north pole of a sphere can be parallel if moved on the sphere to the equator.

We can describe the upper hemisphere of a unit sphere by the equation $z = \sqrt{1 - x^2 - y^2} = \sqrt{1 - r^2}$. Then we can look at the tangent vectors of the surface. At the north pole, we can describe a tangent plane with the vectors $\hat{x}$ and $\hat{y}$, like the ones in the image above. But if we take these same tangents and move them parallel to the direction they are pointing, then bring one over to the other, something strange happens...they are now the same vector. We aren't describing a tangent plane, we are only describing one tangent direction. So while the nice, orthogonal (x,y) coordinate system works on one point of a sphere, that coordinate system stops working correctly if you move too far away. One way to see this is to write down the metric for the upper hemisphere in cartesian coordinates[10]:

$$h_{ij}(x,y) = \begin{pmatrix} \frac{1-y^2}{1-x^2-y^2} & \frac{xy}{1-x^2-y^2} \\ \frac{xy}{1-x^2-y^2} & \frac{1-x^2}{1-x^2-y^2} \end{pmatrix}_{ij} \tag{2.3.3}$$

[10]Can you see how to get this metric representation?

When we look at the North pole (x,y) = (0,0), then this metric is just $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and it looks like our nice cartesian description of the plane $\mathbb{R}^2$. This means that locally, it is "flat." But as soon as we move away from this point, we get a bunch of off diagonal terms, and the diagonal ones aren't normalized properly. To put this in the language of general relativity, for small distances and times (looking locally), spacetime should be "flat," and look like special relativity, but if we wait too long, or look over too large distances, we'll see some sort of acceleration, no matter what coordinates we use. It's this "apparent acceleration" that we call gravity. In reality, it's just the fact that spacetime is curved, so we can't write "straight" trajectories in any coordinate system (without adding non-gravitational forces). But looking at the sphere, it looks like we might have to be careful about following straight lines or tangent vectors, since two vectors that start out orthogonal seem to become parallel after moving them around the sphere. If we are going to describe motion through curved geometries, we'll clearly have to be more careful.

2.4. **Moving on curved surfaces.** *Warning: The next two sections will have bits with a lot of math that won't make intuitive sense until near the end. If you're confused, just keep reading until the end of both sections, and see if things clear up a bit.*

The example of motion along the surface of a sphere shows that when we talk about moving on a curved surface, we have to be very careful of the coordinates we are using to describe the motion. If we aren't, the coordinates we are using can stop being orthogonal, and even turn enough to point in the same direction! The solution to all of this? Use multiple different coordinates for different patches of a surface! This should already be familiar from using cartesian coordinates to describe conic sections: circles, hyperbolas, ect. In these cases, you can only ever describe one "branch" of the curve at a time with your usual (x,y) coordinates. To describe the other branch, you need to transform your coordinates (usually by taking new coordinates $y' = -y$).[11] This should be enough to begin describing motion on curved surfaces. We know that at a point, we can use coordinates that look like "flat" spacetime, $\bar{x}^\mu$ (from here on, I'll try to use bars to indicate local/flat coordinates), but we can transform these coordinates into more general, curved ones, x^μ. Recall the condition that spacetime coordinates were "flat" was simply that their second derivative with respect to τ vanished:

$$\frac{d^2\bar{x}^\mu}{d\tau^2} = 0 \tag{2.4.1}$$

Well if we transform velocities in this "flat" frame $\bar{u}^\mu = \frac{d\bar{x}^\mu}{d\tau}$ into curved coordinates as above: $u^\mu = \frac{dx^\mu}{d\tau} = \frac{d\bar{x}^\nu}{d\tau}\frac{\partial x^\mu}{\partial \bar{x}^\nu} = \bar{u}^\nu \frac{dx^\mu}{d\bar{x}^\nu}$ and take derivatives carefully, we can see what

[11] An important addition is that the transformation must match at any points of overlap (like y = y' = 0 above) as must their derivatives at overlapping points.

the "acceleration" of these straight trajectories will look like in these more general coordinates[12]:

$$\begin{aligned} \frac{d^2x^\mu}{d\tau^2} &= -\Gamma^\mu_{\nu\rho}\frac{d\bar{x}^\nu}{d\tau}\frac{d\bar{x}^\rho}{d\tau}, \quad \text{or} \\ \frac{du^\mu}{d\tau} &= -\Gamma^\mu_{\nu\rho}u^\nu d\tau u^\rho, \quad (\text{where } \Gamma^\mu_{\nu\rho} = \frac{\partial x^\mu}{\partial \bar{x}^\sigma}\frac{\partial^2 \bar{x}^\sigma}{dx^\nu dx^\rho}) \end{aligned} \tag{2.4.2}$$

There are a few important things to point out about this funny term called a **Christoffel symbol**: $\Gamma^\mu_{\nu\rho}$. The first, is that it is explicitly coordinate dependent. This means that, much like the matrix *representation* of a metric, if you change coordinates, then the Christoffel symbol will be different.[13] The second thing to notice is that all Christoffel symbols should vanish at the origin for "good" coordinate systems (ones that explicitly satisfy the equivalence principle such that spacetime looks flat at the origin). But for a general curved background, you can only get the symbols to vanish at one point, and no matter how you transform your coordinates, they won't be zero over any finite region of spacetime. Finally, these Christoffel symbols seems to act like a potential for gravity. On the left side of the equation above, we have an acceleration while on the right, we have these funny symbols telling us about how curved spacetime is. This should remind you of the equation for the non-relativistic equation for force:

$$\vec{F}_{grav} = \frac{d\vec{p}}{dt} = m\frac{d\vec{u}}{dt} = -m\nabla\Phi_{grav} \tag{2.4.3}$$

Where Φ_{grav} is the gravitational potential energy for a distribution of matter ($-\frac{GM}{|\vec{r}|}$ for a point mass M at the origin). Looking at Eq. (2.4.2), we see this is the first instance where we can start saying that the "force" of gravity is nothing but the effects of spacetime being curved.[14] Well now we know how initially "striaght" trajectories look on a curved surface. Now let's look at initially straight velocities or tangent vectors. If we have a 4-vector that is straight or constant in flat spacetime (special relativistic coordinates $\bar{x}^\mu$), then we know that it shouldn't change orientation or length regardless of where we move it (I can take the unit vector pointing up at the origin of the plane $\mathbb{R}^2$ given by $\hat{y}$, and move it around to any point in the plane, but it will still point up with the same magnitude):

$$\frac{\partial v^\mu}{\partial \bar{x}^\nu} = 0 \tag{2.4.4}$$

[12] Showing this is a good exercise, but can be a bit frustrating if you're not comfortable with moving indicies.

[13] There is a bit more subtlety here. The Christoffel symbols actually don't transform like a tensor, which means you can't just hit them with chain rule factors of $\frac{\partial x^\mu}{\partial y^\nu}$ to change coordinates from x to y.

[14] More precisely, at this point, we only know that this effect comes from spacetime not being flat, but we will make the definition of curvature more concrete in future.

Now, it should come as little surprise that this will not be true in more general coordinates x^ν. In this case:

$$\frac{\partial v^\mu}{\partial x^\nu} = -\Gamma^\mu_{\nu\rho}\frac{dx^\rho}{d\tau} = -\Gamma^\mu_{\nu\rho}u^\rho \tag{2.4.5}$$

All this is saying is that vectors that look constant in one frame may get twisted in the direction of u^ρ if you move them along the path given by x^ν. This is exactly the reason our orthogonal coordinate system on the north pole of a sphere could be reduced to one direction on the equator. The vectors $\hat{x}$ and $\hat{y}$ got twisted as they moved down to the equator then over to meet each other until they were parallel vectors! We'll go through this exact example in a little while, but before we do, we need to develop a little more machinery. The first is a new linear operator that defines when an object is moving in a "constant" direction along a curved surface. This is exactly the situation described above, so if we just move the terms on the right hand side over, we can define something called the **covariant derivative**: ∇_μ.

$$\nabla_\nu v^\mu = \frac{\partial v^\mu}{\partial x^\nu} + \Gamma^\mu_{\nu\rho}u^\rho = 0 \tag{2.4.6}$$

Here, the covariant derivative is zero because we acted on the appropriate "constant" velocity vector. Just like a regular derivative, it's usually not zero for an arbitrary function, but when $\nabla_\nu v^\mu = 0$, we call the velocity, v^μ, "covariantly constant." Notice, however, that this derivative is much more like a gradient: ∇_μ is actually a collection of four derivatives (one for each index). Now we can take this one step further by focussing on the derivative in a definite direction (focussing on a component of the gradient). If we pick a direction given by a 4-vector u^μ, then we can define something called the **absolute derivative**:

$$\frac{Dv^\mu}{d\tau} = u^\nu\nabla_\nu v^\mu = \frac{dx^\nu}{d\tau}\frac{\partial v^\mu}{\partial x^\nu} + \Gamma^\mu_{\nu\rho}u^\rho u^\nu = \frac{dv^\mu}{d\tau} + \Gamma^\mu_{\nu\rho}u^\rho u^\nu \tag{2.4.7}$$

The absolute derivative tells us about how a single vector will look as we follow the path given by u^ρ. And in the spirit of what we did above, if the covariant derivative is zero, the absolute derivative must be zero as well. So now we have a condition to move around a single "locally constant" 4-vector along a curved surface. This proceedure is called **Parallel transport**, and quantitatively, it is given by:

$$\frac{Dv^\mu}{d\tau} = \frac{dv^\mu}{d\tau} + \Gamma^\mu_{\nu\rho}u^\rho u^\nu = 0 \tag{2.4.8}$$

If we specialize even further, and instead of choosing just any "locally constant" 4-vector, we choose the one corresponding to the tangent of a path followed in spacetime (u^μ), then what we get is the equation of "straight" (meaning unaffected by no outside forces) motion through spacetime:

$$\frac{Du^\mu}{d\tau} = \frac{du^\mu}{d\tau} + \Gamma^\mu_{\nu\rho} u^\nu u^\rho = 0 \tag{2.4.9}$$

Equivalently, we can express this in terms of the worldline of a particle $x^\mu(\tau)$, feeling no forces:

$$\frac{D}{d\tau}\left(\frac{dx^\mu}{d\tau}\right) = \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0 \tag{2.4.10}$$

Which just reproduces Eq. (2.4.2), describing the motion of a particle in curved coordinates. We call such a "straight" path that has no external forces pushing it off its natural trajectory a **geodesic**. Framed in this way, gravity is just an extension of Newton's 1st(?) law: in the absence of external forces, objects travel along "straight" lines (geodesics) at "locally constant" velocities ($u^2 = -1$). Hopefully this is starting to make some sense, but to really get a feel for these concepts, you need to calculate something. So lets get down to explicitly calculating Christoffel coefficients (symbols).

2.5. **Calculating Christoffel coefficients.** It turns out that the equivalence principle is enough to allow us to write Christoffel coefficients in terms of our spacetime metric, $g_{\mu\nu}$. Proving this isn't too difficult, but it is a long calculation moving a lot of indicies. Essentially, all you need is to say that the dot product of two locally constant vectors should be constant along a geodesic if the equivalence principle holds, so[15]

$$0 = \frac{D}{d\tau}(v \cdot w) = \frac{D}{d\tau}(v_\mu w_\nu g^{\mu\nu}) = v_\mu w_\nu \left(u^\sigma \left(\frac{\partial g^{\mu\nu}}{\partial x^\sigma} + \Gamma^\mu_{\sigma\rho} g^{\sigma\nu} + \Gamma^\nu_{\sigma\rho} g^{\sigma\mu}\right)\right)$$

Working through some annoying algebra, one can use this to show that the Christoffel coefficients can be written in terms of metric coefficients as:

$$\boxed{\Gamma^i_{jk} = \frac{1}{2} g^{il}(\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk})} \tag{2.5.1}$$

Now let's figure out what these coefficients look like for motion on the surface of a sphere. We'll do this in two coordinate systems to show how different the christoffel symbols can be. We'll try the coordinates (θ, ϕ) and polar coordinates (r, θ) The metric is[16]:

$$h_{ij}(\theta,\phi) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}, \quad h_{ij}(r,\theta) = \begin{pmatrix} \frac{1}{1-r^2} & 0 \\ 0 & r^2 \end{pmatrix}, \tag{2.5.2}$$

[15] This employs the covariant (technically absolute) derivative of a rank-2 tensor, which is just like the one we did with vectors, but there is another Christoffel symbol to cancel out the extra index. Can you see how to generalize this for the covariant derivative of a rank-n tensor?

[16] Can you see how to get these?

So the Christoffel coeeficients should be easy enough to calculate[17]. First, in the (θ, ϕ) coordinates, the only non-zero entries are

$$\Gamma^{\phi}_{\phi\theta} = \Gamma^{\phi}_{\phi\theta} = \cot\theta, \quad \Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta, \tag{2.5.3}$$

Next, in the (r, θ), coordinates the only non-zero entries are:

$$\Gamma^{\theta}_{\theta r} = \Gamma^{\theta}_{r\theta} = \frac{1}{r}, \quad \Gamma^{r}_{rr} = -\frac{r}{1-r^2}, \quad \Gamma^{r}_{\theta\theta} = r^3 - r \tag{2.5.4}$$

With these in hand, one can actually calculate how the coordinate vectors in Fig. 2.1 shift to be parallel. This is a bit of a lengthy calculation, so we'll skip it here, and move back to GR-related material. Let's look to see how Christoffel coefficients can give us gravitational forces in the no-relativistic limit. To see this, we'll want to carefully choose the conditions that give us a non-relativistic limit. We'll state the conditions with brief explanations, but it's very worthwhile to think about why these make sense:

(1) Small velocities: $(v << c = 1)$. This should be familiar from SR. In this limit, we can start treating proper time like regular time since $\gamma \approx 1$
(2) Small corrections to the metric: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $|h_{\mu\nu}| << 1$. This means that spacetime looks *almost* like it does in SR, so all gravitational influences, coming from the non-trivial $h_{\mu\nu}$ term look like weak forces on a flat background spacetime.
(3) Slow changes: $\partial_0 g_{\mu\nu} = \partial_0 h_{\mu\nu} \approx 0$. This just means that the gravitational influences aren't changing in time too fast relative to the speed of light.
(4) Moderate gradients: $\partial_i g_{\mu\nu} = \partial_i h_{\mu\nu} =$ "small"(ϵ). This just means that the gravitational influences aren't fluctuating too much over space (like near a black hole)

With these assumptions, we should be able to figure out a non-relativistic "force" of gravity. To do so, let's recall our geodesic equation:

$$\frac{d^2x^{\mu}}{d\tau^2} = -\Gamma^{\mu}_{\nu\rho}\frac{d\bar{x}^{\nu}}{d\tau}\frac{d\bar{x}^{\rho}}{d\tau}$$

This looks a great deal like acceleration, so if we want to look at a non-relativistic force, we'll want to look at something like $F^i = m\frac{d^2x^i}{dt^2}$. Luckily, in our since $v << c$, $\tau \approx t$. And we can also use this to say that $\frac{d\bar{x}^{\rho}}{d\tau} \approx (1, \frac{\vec{v}}{c}) = (1, \vec{\epsilon})$, where $\vec{\epsilon}$ is just a tiny vector. Using all of this, and focusing on the spacial components of the left hand side, we have:

$$F^i_{grav} = m\frac{d^2x^i}{d\tau^2} \approx m\frac{d^2x^i}{dt^2} = -m\Gamma^i_{\nu\rho}u^{\nu}u^{\rho} \tag{2.5.5}$$

So all we need to do to figure out the gravitational force is calculate these Christoffel symbols and substitute in the values of the 4-velocities, u^{μ}. But this is still the sum of a lot of components, so let's see if any of the terms is small enough to ignore. At this point,

[17]remember that h^{ij} is the inverse of the metric $h_i j$, so that for the plane, $\mathbb{R}^2$, $\delta^{ij}(r,\theta) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}$

one might be tempted to use the fact that spacial components of the 4-velocies, $\vec{\epsilon}$ is so small to start ignoring those components compared to the time-like components that are just 1. But to be careful, we need to check that the corresponding Christoffel symbols are small too:

(2.5.6)

$$\begin{aligned}\Gamma^i_{\nu\rho} &= \frac{1}{2}g^{i\sigma}(\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho}) = \frac{1}{2}(\eta^{i\sigma} + h^{i\sigma})(\partial_\nu h_{\rho\sigma} + \partial_\rho h_{\nu\sigma} - \partial_\sigma h_{\nu\rho})\\ &\approx \frac{1}{2}\eta^{i\sigma}(\partial_\nu h_{\rho\sigma} + \partial_\rho h_{\nu\sigma} - \partial_\sigma h_{\nu\rho}) = \frac{1}{2}(\partial_\nu h_{\rho i} + \partial_\rho h_{\nu i} - \partial_i h_{\nu\rho})\\ &= \frac{1}{2}(2\partial_0 h_{0i} - \partial_i h_{00} + \partial_0 h_{ki} + \partial_k h_{0i} - \partial_i h_{0k} + \partial_j h_{0i} + \partial_0 h_{ji} - \partial_i h_{j0} + \partial_j h_{ki} + \partial_k h_{ji} - \partial_i h_{jk})\\ &\approx \frac{1}{2}(-\partial_i h_{00} + \partial_k h_{0i} - \partial_i h_{0k} + \partial_j h_{0i} - \partial_i h_{j0} + \partial_j h_{ki} + \partial_k h_{ji} - \partial_i h_{jk})\end{aligned}$$

In the first step, we used the definition of $g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}$ and the fact that $|h_{\mu\nu}| << 1$ and $\eta^{\mu\nu}$ is constant to get to the second line. From there, we know that the spacial components of the Minkowski metric are diagonal and equal to $\eta^{ij} = 1$. After that, we simply expand out all of the terms in the sum and drop all of the ones we said were zero in the Non-relativistic limit. At this point, since all of the terms are of the form $\partial_i h_{\mu\nu}$, all we know is that all of these terms are similarly small. At this point, we can now say that only the $(\nu\rho) = (0,0)$ terms survive because their 4-velocity components "1", are much bigger than the other components $\vec{\epsilon}$. Thus:

$$F^i_{grav} = -m\Gamma^i_{00} \approx \frac{1}{2}(\partial_i h_{00}) = -m\partial_i \Phi_{grav} \tag{2.5.7}$$

Where we used the definition Newtonian gravitational potential energy on the right. From this, we learn two things. First, the gravitational force comes primarily from the gradient of the (0,0) component of the metric, $g_{0,0}$. Second, since we know what the force is for Newtonian gravitation, we can figure out what this component looks like for a point mass, M:

$$g_{00} = \eta_{00} + h_{00} = -(1 + 2\Phi_{grav}) = -\left(1 - \frac{2GM}{R}\right) \tag{2.5.8}$$

This component is actually the exact right answer for the time-like (0,0) component of the spacetime metric with a mass, M, in empty space. This metric, which also describes a stationary black hole (more on that later) is given by a spacetime interval

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\left(1 - \frac{2GM}{R}\right)dt^2 + \frac{dr^2}{(1 - \frac{2GM}{R})} + R^2 d\theta^2 + R^2 \sin^2\theta d\phi^2 \tag{2.5.9}$$

Notice that when M = 0, this just gives us the regular minkowski metric $\eta_{\mu\nu} = diag(-1,1,1,1)$. This is also the case infinitely far away from the mass ($R \to \infty$). When either is finite, this is not the case. This allows for a really interesting phenomenon: *gravitational time-dilation*. If we look at the proper time for a fixed point infinitely far away ($d\tau = dt$), and compare it to the proper time from the reference frame of a fixed point at radius R away from the mass $\left(d\tau^2 = \left(1 - \frac{2GM}{R}\right)(dt')^2\right)$, we find something eerily similar to time-dilation in special relativity:

$$dt' = \frac{dt}{\sqrt{1 - \frac{2GM}{R}}} \tag{2.5.10}$$

This means that the closer one is to the mass, M, the more intense the gravity, the more time slows down. So now we see that both intense gravity and high relative velocities cause time dilation! This purely general relativistic effect is actually significant enough that it has to be taken into effect when GPS satelites triangulate positions using light signals.

3. Curvature, Matter, and Einstein's Field Equations

Now that we know a bit about differential geometry, we can start doing a bit more physics. From what we did above, it should be clear that the "gravitational forces" are just the result of following geodesics, or "straight" trajectories, in a curved spacetime. But if we want to quantify this "force," we should do so in a relativistic manner: the measure of how "curved" a spacetime is shouldn't depend on the coordinate system we use. For this reason, we'll start moving away from the coordinate-dependent Christoffel symbols and instead describe something called the **curvature tensor**, $R^{\rho}_{\sigma\mu\nu}$, which describes how much a vector changes direction after going around a small area $dx^{\mu}dx^{\nu}$ (or alternatively, the difference in the vector after moving along a path dx^{μ} then path dx^{ν} vs dx^{ν} first, then dx^{μ}). See figure 3.1 for intuition.

Mathematically, this means that parallel transport is non-commutative (order matters). And can be represented by using our covariant derivatives along the paths described:

$$[\nabla_{\mu}, \nabla_{\nu}]v^{\rho} = R^{\rho}_{\sigma\mu\nu}v^{\sigma} \tag{3.0.1}$$

This can be expressed in terms of Christoffel symbols as follows (remember though, the curvature is actually coordinate independent):

$$R^{\rho}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\rho}_{\nu\sigma} - \partial_{\nu}\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\xi}\Gamma^{\xi}_{\nu\sigma} - \Gamma^{\rho}_{\nu\xi}\Gamma^{\xi}_{\mu\sigma} \tag{3.0.2}$$

The important thing is that, like Newton's equation's these are second-order derivatives going into the final law. Thus, the information needed to solve the differential equation we will soon write down will need analogues of both initial "position" and "velocity".[18] To make this point more clear, let's re-write the curvature tensor with all lower indicies.[19]

[18]Can you guess what these should be in the case of worldlines in GR?

[19]It is a good exercise to see if you can reproduce this result. It isn't hard, but you need to be careful to keep track of indicies

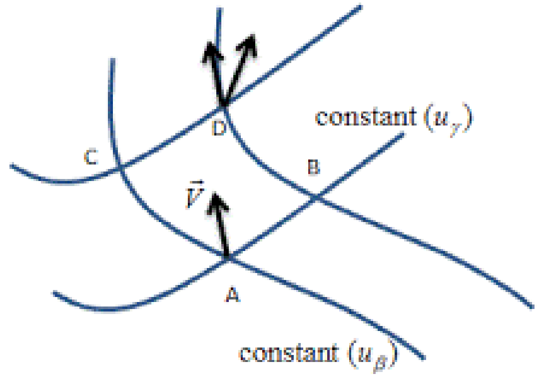

Fig 1.2

FIGURE 3.1. Curvature visualized as a difference in the direction of a vector being parallel transported along two paths (direction γ in the a-B direction and direction β in the D-A direction) in different orders. ($A \to B \to C$ vs $A \to D \to C$)

$$R_{\rho\sigma\mu\nu} = g_{\rho\xi} R^{\xi}_{\sigma\mu\nu} = -\frac{1}{2}\left(\partial_\rho\partial_\mu g_{\sigma\nu} - \partial_\rho\partial_\nu g_{\sigma\mu} + \partial_\sigma\partial_\nu g_{\rho\mu} - \partial_\sigma\partial_\mu g_{\rho\nu}\right) \tag{3.0.3}$$

This is the most detailed account of curvature, but if we contract with the metric, we can get a coarser description of the curvature called the **Ricci curvature** (tensor), $R_{\mu\nu}$

$$R_{\mu\nu} = g^{\rho\sigma} R_{\rho\mu\sigma\nu} = R^{\rho}_{\mu\rho\nu} \tag{3.0.4}$$

And an even more coarse description can come in the form of a **scalar curvature**, R

$$R = g^{\mu\nu} R_{\mu\nu} \tag{3.0.5}$$

This just gives a general idea about whether the curvature is positive or negative. To get an idea for this curvature, let's try calculating these quantities for a sphere, whose metric in angular coordinates is simply $g_{ij} = R^2 \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$

Before diving into calculation, let's think about which components ought to be zero. Well, bu symmetry, we expect the all-θ and all-ϕ components to vanish. Likewise, the metric is diagonal and has no ϕ-dependence, so any components with just one or three ϕ indicies will vanish. Thanks to the symmetries of the curvature tensor, this actually leaves us in pretty good shape. Now we only need to calculate components with two θ and two ϕ indicies. We already have the Christoffel coefficients from a calculation above:

$$\Gamma^{\phi}_{\phi\theta} = \Gamma^{\phi}_{\theta\phi} = \cot\theta, \quad \Gamma^{\theta}_{\phi\phi}$$

From this, we can easily get the only non-zero element (up to symmetry) of the curvature tensor

$$\begin{aligned} R^{\theta}_{\phi\theta\phi} &= \partial_\theta \Gamma^{\theta}_{\phi\phi} - \partial_\phi \Gamma^{\theta}_{\phi\theta} + \Gamma^{\theta}_{n\theta}\Gamma^{n}_{\phi\phi} - \Gamma^{\theta}_{n\phi}\Gamma^{n}_{\phi\theta} \\ &= -\cos 2\theta + \cos^2\theta = \sin^2\theta \end{aligned} \tag{3.0.6}$$

This makes getting the Ricci tensor really easy:

$$R_{ij} = \frac{1}{R^2} g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \tag{3.0.7}$$

And finally, contracting with the metric makes the scalar curvature $R = g^{ij}R_{ij} = \frac{2}{R^2}$. And, as might be expected, this curvature is positive, uniform, and proportional to $1/r^2$ for a sphere. With these intuitions, let's look at how curvature fits with mass-energy by looking at the equation(s) that made Einstein accomplishments in GR really shine through: Einstein's field equations (EFE) are given by:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{3.0.8}$$

The left-hand side has to do with curvature (second derivatives of the metric)[20], while the right-hand side has to do with the matter content of spacetime. If we recall the definition of $T_{\mu\nu}$:

$$T^{\mu\nu} = \frac{\delta p^\mu}{\delta^4 x} \delta x^\nu$$

.

So that the 00 component is simply energy density: $T^{00} = \frac{\delta p^0}{\delta^3 x} = \frac{Energy}{Volume}$. It is easy to check that the others are momenta density, pressure densities, and stress/strain densities. But more importantly, since the EFE relate spacetime and matter so directly, we can see that if $\partial_\sigma(g_{\mu\nu}) = 0$, then $\partial_\sigma(T^{\mu\nu}) = 0$ and there is a conserved quantity associated with the direction K^σ.

a constant **Killing vector** is a vector that satisfies $K^\sigma \partial_\sigma(g_{\mu\nu}) = 0$. This implies a conserved quantity $K_\mu p^\mu = g_{\mu\nu} K^\mu p^\nu$. As an example, if there is no time-dependence in the metric, then $\hat{t}^0 \partial_0(g_{\mu\nu}) = 0$ and we would expect a conserved quantity (which we will label with E for reasons that will become apparent):

$$E = -g_{00}\hat{t}^0 p^0 = -g_{00}\frac{dt}{d\tau} = -\gamma m g_{00} \tag{3.0.9}$$

[20]The Λ term is one that essentially adds inherent curvature to spacetime. Alternatively, we could move it to the "matter" side, where it will be negative, and act like "repulsive" mass. This is the term responsible for the acceleration of expansion in our universe and represents the concentration of "dark energy" in our universe

Where $\gamma = \frac{1}{\sqrt{1-v^2}}$ is the lorentz factor from time dilation in the special relativity section. The negative sign is simply a convention to make sure this "energy" is positive. If we take a familiar metric, the minkowski metric of flat spacetime, then $g_{00} = \eta_{00} = -1$, and thus $E = \gamma m$, which is exactly the formula for energy in SR! So this formula simply generalizes what energy is in curved spacetimes that admit of time-independent metrics. The same can be done for any other coordinates that do not appear in the metric, like ϕ, which will give a conserved quantity of angular momentum (makes sense, right?).

4. Black Hole metrics

Now as you may have guessed from the frustrating number of indicies and differential objects above, Einstein's equations are *really* hard to solve. So much so that there are only a handful of analytic solution. Luckily, black holes have enough symmetries to produce such "nice" solutions. Black holes are very special objects. They warp spacetime so much that they pull all matter that crosses their event-horizon into a singularity. Near this point, the gravity is actually so strong in such small regions of spacetime that quantum effects become important and GR breaks down. But outside the singularity (where we actually understand what's going on), the only affects on spacetime come from long-range properties of the matter in the singularity.[21] Imagine a point particle. If all you care about are gravity and electromagnetism, you can specify pretty much everything about this particle's dynamics once you know its mass, m, a charge, q, and an intrinsic spin, s (or angular momentum). So let's see what we can understand about black holes by considering the effects of these properties.

4.1. Schwarzchild Black Hole.

First, let's look at the metric for a stationary black hole of mass, M. This is called the Schwarzchild metric[22]:

$$d\tau^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2GM}{r}\right)} + r^2 d\Omega^2 \tag{4.1.1}$$

Where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the differential of a solid angle (region on a sphere). This has a few point of interest. First, if we go off infinitely far from the black hole ($r \to \infty$), then the metric becomes minkowskian $d\tau^2 = -dt^2 + dr^2 + r^2 d\Omega^2$, as we should hope it would. Next, there seem to be two problematic radii: $r = 0$ and $r_s = 2GM$. These both produce singularities in the metric in these coordinates. It turns out that only the former singularity is fundamental. One can calculate the intrinsic, coordinate independent curvature at $r = 0$, and see that it blows up so we know GR has failed us. But when $r = 2GM$, we can actually get rid of the singularity by picking better coordinates.[23] But something interesting does happen past the event horizon ($r < r_s = 2GM$). Here, we see that the time coordinate

[21] We'll go into more detail in the next section when discussing the 0th law of Black Hole Thermodynamics.

[22] You may have noticed this is actually an equation for the proper time, but you should be able to extract the metric from this pretty easily by now

[23] See Kruskal ? Szekeres coordinates for such an example.

and radial coordinate flip sign, so that it seems they have interchanged roles. And in some sense, they have. But this there is no mystical interchange of time and space and some may wish to claim. Rather there is a subtle shift in cause and effect. In normal experience, we cannot help but move forward in time, regardless of how we move through space. However, as soon as one passes the event horizon, there is an extreme shift in one's light cone (which shows what can be causally connected to a point), such that moving towards the singularity is inevitable no matter how one "moves through time." Further, since the metric doesn't depend on the time coordinate or the azimuthal coordinate ($\partial_t(g_{\mu\nu}) = \partial_\phi(g_{\mu\nu}) = 0$), we expect two conserved quantities corresponding to energy and angular momentum:

$$\begin{aligned} E &= g_{\mu\nu}\hat{t}^\mu p^\nu = -g_{t\nu}p^\nu = m\left(1 - \frac{2GM}{r}\right)\frac{dt}{d\tau} \\ L &= g_{\mu\nu}\hat{\phi}^\mu p^\nu = g_{\phi\nu}p^\nu = 0 \end{aligned} \tag{4.1.2}$$

Where we have to be careful to note that p^μ is a linear 4-momentum, and thus, has no angular component. As expected, no angular momentum and an energy that depends on the proximity to the black hole with some weird stuff happening near the event horizon. It seems like energy is negative when $r < 2GM$, but this is kind of ok, since we can't really interact with anything past the event horizon, so it's not as though we could use "negative energies" to fuel infinite energy outside the event horizon. You can kind of think of this as simply redefining the sign associated with the "time" coordinate, just like in the metric. There's still a conserved energy, but it has to be a different sign from whatever convention we choose outside the event horizon as a consequence of the weird causal disconnect past the horizon.

Let's discuss another important quantity: **surface gravity** is akin to the acceleration felt at an object's surface. On the earth's surface, the surface gravity is $g = \frac{GM}{R^2} \approx 9.81 m/s^2$. Unsurprisingly, defining accelerations in GR is a bit more complicated, but the intuition holds surprisingly well. In GR, it is defined as the acceleration, seen from an observer infinitely far away, needed to keep an object on the designated surface. If K^μ is a killing vector (generates a conserved quantity), then the surface gravity, κ mathematically is defined by

$$K^\mu \nabla_\mu K^\nu = \kappa K^\nu \tag{4.1.3}$$

There are various tricks to calculating these, but for now, I will simply quote the result for a Schwarzchild black hole: $\kappa = \frac{1}{4GM}$. This kind of makes sense since the event horizon is at $R_S = 2GM$, so that $g = \frac{GM}{R_S^2} = \frac{1}{4GM}$.

4.2. **Kerr Metric.** Now let's look at a metric for a rotating black hole with mass M and angular momentum J:

(4.2.1)
$$d\tau^2 = \left(1 - \frac{R_S r}{\Sigma}\right) dt^2 - \left(\frac{\Sigma}{\Delta}\right) dr^2 - \Sigma d\theta^2 + \left(r^2 + a^2 + \frac{R_S r a^2 \sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2 + \frac{2R_s r a \sin^2\theta}{\Sigma} dt d\phi$$

Where we have made the following substitutions for convenience: $R_S = 2GM$, $a = \frac{J}{GM}$, $\Sigma = r^2 + a^2\cos^2\theta$, and $\Delta = r^2 - 2GMr + a^2$. This metric imposes a few limits on these quantities[24]: $|a| \leq 1$, $\Sigma \geq r^2$. If we look very closely at this metric, we see that problems start to appear if either $\Delta = 0$ or $\Sigma = 0$. For the former case, we find two different radii:

$$r_\pm = GM \pm \sqrt{G^2M^2 - a^2} \tag{4.2.2}$$

A little algebra shows that at the outer radius, r_+, the purely time component becomes:

$$-\frac{1}{\Sigma}(\Sigma - R_S r) dt^2 = -\frac{1}{\Sigma}\left(\Delta - a^2\sin^2\theta\right) dt^2 = \frac{a^2\sin^2\theta}{\Sigma} dt^2 \geq 0$$

Thus, the time coordinate has already flipped to be positive, and we expect that the same will be true of energy. In this case, however, flipping of the time component does *not* mean that nothing can escape this region. Instead, the causal flip is that nothing can rotate in the opposite direction of the black hole in this region. It is actually the region within the second radius, r_-, where nothing can escape out to infinity. But let's look back at this odd negative energy region within the outer radius. Again, since the metric doesn't depend on the time coordinate or the azimuthal coordinate, we get our conserved quantities by looking at vectors in their directions:

$$\begin{aligned} E &= -g_{t\nu}p^\nu = m\left(1 - \frac{2GMr}{\Sigma}\right)\frac{dt}{d\tau} + \frac{2GMmar}{\Sigma}\sin^2\theta\frac{d\phi}{d\tau} \\ L &= g_{\phi\nu}p^\nu = -\frac{2GMmar}{\Sigma}\sin^2\theta\frac{dt}{d\tau} + \frac{m(r^2+a^2)^2 - m\Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta\frac{d\phi}{d\tau} \end{aligned} \tag{4.2.3}$$

These seem fairly complicated, but we can focus on some of the broader qualities. The first thing to note is that the first term is the same as g_{00}, and thus is negative within r_+. So it does not take much imagination (or too much algebra) to show that there are configurations where the energy is negative without crossing into the final event horizon $r < r_-$. Because of this, it is actually possible to extract energy from a spinning black hole by throwing things in! The fix to this is that as you extract energy, you actually slow down the black hole's rotation. Take a look at L and you'll see that adding negative energy to the black hole requires adding angular momentum against its rotation, and so you can only continue to extract energy until it stops rotating. No infinite free energy for us :(

Now we turn to the surface gravity. As before, I will simply quote the result for the surface gravity for a Kerr black hole: $\kappa = \frac{1}{4GM} - M\Omega_+^2$, where Ω_+ is the angular velocity at the event horizon. The important thing to note here is that $\kappa \to 0$ when Ω_+ and

[24]Can you see why these should hold? It's actually a bit subtle

consequently J gets large enough. This will be significant for the 3rd law of Black Hole Thermodynamics. Now let's add some charge in.

4.3. **Reissner Nordstrom metric.** Charge is a tough thing to describe in GR. Technically, to do it correctly, you don't salve Einstein's field equations. You have to solve a more general set of equations that turns classical electromagnetism into geometry as well. The cost of doing this is introducing a "wrapped up" 5th dimension which accounts for the gauge symmetry in E&M. Roughly speaking, positive charges correspond to mass spinning around this extra dimension in one polarity, while negative charges spin in the other polarity. But when you reduce down to 4 dimensions, you can still describe the same kind of dynamics, but now, the charges just become scalar quantities. And luckily, for black holes, the charge is conserved! We'll skip over any further detail for now, and just push on to the metric

$$d\tau^2 = \left(1 - \frac{R_S}{r} + \frac{R_Q^2}{r^2}\right) dt^2 - \left(1 - \frac{R_S}{r} + \frac{R_Q^2}{r^2}\right)^{-1} dr^2 - r^2 d\Omega^2 \tag{4.3.1}$$

Where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$, $R_S = 2GM$, and $R_Q^2 = \frac{Q^2 G}{4\pi\epsilon_0}$. As with the Kerr-solution, there are actually two horizons, given by the solutions to $1 - \frac{R_S}{r} + \frac{R_Q^2}{r^2} = 0$. This gives solutions:

$$r_\pm = \frac{1}{2}\left(R_S \pm \sqrt{R_S^2 - 4R_Q^2}\right) \tag{4.3.2}$$

r_+ indicates the event horizon, while r_- is the Cauchy horizon. A little algebra shows that the time component changes sign at r_+ (try plugging in $r = \frac{R_S}{2}$) but unlike the case of the Kerr-solution, once you enter the outer horizon (the event horizon), you already can't escape The inner horizon has very technical significance that basically amounts to the fact that inside this horizon, it looks like there exist closed time-like curves (time loops). But what this means physically is far from clear, so we'll just move on. Now what goes on Let's look at the conserved quantities. As we said before, the charge is conserved, but deriving this conservation takes us too far into Einstein-Maxwell theory for these notes, so we'll just cite the conserved charge coming from the electromagnetic potential of the black hole $A_\mu = (Q/r, 0, 0, 0)$, and move onto the energy and angular momentum:

$$\begin{aligned} E &= g_{\mu\nu}\hat{t}^\mu p^\nu = -g_{t\nu}p^\nu = m\left(1 - \frac{R_S}{r} + \frac{R_Q^2}{r^2}\right)\frac{dt}{d\tau} \\ L &= g_{\mu\nu}\hat{\phi}^\mu p^\nu = g_{\phi\nu}p^\nu = 0 \end{aligned} \tag{4.3.3}$$

From the discussion above, we see that energy can be negative inside the event horizon. But like the negative energy for the Schwarzchild black hole, this shouldn't concern us, as it is disconnected from any measures of energy outside the event horizon.

Finally, let's look at the surface gravity. As before, I will simply quote the result for the surface gravity for a charged black hole: $\kappa = \frac{\sqrt{R_S^2/4 - R_Q^2}}{R_S^2/2 - R_Q^2 + R_S\sqrt{R_S^2/4 - R_Q^2}}$, where R_S and R_Q are, as defined above, quantities proportional to the mass and charge respectively. The important thing to note here is that $\kappa \to 0$ when $R_Q \to 2R_S$ and consequently Q gets large enough. This will also be significant for the 3rd law of Black Hole Thermodynamics. Now that we've discussed angular momentum and charge independently, we'll just write down what the metric looks like when everything is put together:

4.4. **Kerr-Newman metric.** Because of how complicated this metric is, we'll be expressing it in a condensed, slightly unfamiliar form. If you want, you can just expand out the compressed quantities to recover a more familiar expression:

$$d\tau^2 = \rho^2 \left(\frac{dr^2}{\Delta} + \delta\theta^2 \right) + \frac{\Delta}{\rho^2} \left(dt - a\sin^2\theta d\phi \right) dr^2 + \frac{\sin^2\theta}{\rho^2} ((r^2 + a^2)d\phi - adt)^2 \tag{4.4.1}$$

Where $a = \frac{J}{M}$, $\rho^2 = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - R_S r + a^2 + R_Q^2$, $R_S = 2GM$, $R_Q^2 = \frac{Q^2 G}{4\pi\epsilon_0}$.

These actually give us 4 horizons by setting the spatial and temporal elements to 0. These give inner and outer event horizons and inner and outer ergospheres.

The surface gravity of this solution can be calculated as $\kappa = \frac{r_+ - r_-}{2(r_+^2 + a^2)}$, where $r_+ = \frac{R_S}{2} \pm \sqrt{\left(\frac{R_S}{2}\right)^2 - R_Q^2 - J^2}$. The important takeaway is that there is a delicate balance between the mass, charge, and angular momentum, so that in the appropriate units, $M^2 \geq Q^2 + J^2$, with equality when the black hole becomes extremal.

5. Black Hole Thermodynamics Primer

Here, I will state the laws of classical thermodynamics in a particularly suggestive way, then quote some results from GR (and some later supplements from QFT in curved backgrounds) as motivation for the laws of Black Hole Thermodynamics (BHT). Now, this field is vast, subtle, and requires a lot of work to show simple things, so as before, this will consist of mostly of undefended claims. A final note, please forgive the numbering system for the laws; it is a historical accident since the 0th law seemed so obvious no one named it till the other three existed and people wanted to systemize thermodynamics.

(1) **0th law**: All systems come to a unique equilibrium, and the macroscopic variables describing the system are homogeneous (Temperature, Pressure, Density, etc are constant through a closed system). This can also be stated as equilibrium being a transitive property (If system A is in equilibrium with system B, and system B is in equilibrium with system C, then system A is in equilibrium with system C)
(2) **1st law**: Energy conservation, $dE = dQ - dW = TdS - PdV - nd\mu - \dots$
(3) **2nd law**: Entropy increase: $\Delta S_{closed} \geq 0$ (the entropy of a closed system increases over finite time scales, meaning there is a tiny chance for fluctuation, but the infrequency of these fluctuations diverges with the system size)

(4) **3rd Law**: Freezing out: As $T \to 0$, $dS \to 0$. (As one approaches absolute zero, there are fewer means to rearrange microstates, and the changes in entropy available tend to zero)

For black holes, there are similar theorems. These require the assumption of Einstein's field equations, some energy conditions amounting to mass being defined as positive in the appropriate manner, and a condition eliminating considerations of "naked singularities" (points of infinite curvature not hidden by an event horizon):

(1) **0th law**: All stationary spacetime solutions are black hole solutions, or more specifically, Kerr-Neuman metrics with the only freedom being in the choice of M, Q, and J. On the event horizon, the surface gravity, κ, is constant
(2) **1st law**: Mass/Energy conservation, $dM = \frac{\kappa}{8\pi} dA - \Omega dJ - \Phi dQ$
(3) **2nd law**: Entropy increase: $\Delta S_{cl} \sim \Delta A_{cl} \geq 0$ (the area of a classical black hole is non-decreasing). Hawking later discovered black holes radiate at temperature $T = \frac{\kappa}{2\pi}$, and thus the entropy of a black hole is $S = \frac{A}{4}$. Thus the generalized entropy law is $\Delta S_{gen} = \Delta S_{BH} + \Delta S_{matter} \geq 0$
(4) **3rd Law**: Freezing out: As $\kappa \to 0$, the black hole becomes extremal. At these points $dA \to 0$.

A final note on why these laws are interesting. In classical thermodynamics, we always have available to us the perspective of statistical mechanics. In this perspective, all of the macroscopic variables we care about in thermo are actually just averages or distributional properties ofstatistical ensembles of microscopic variables. Most pointedly, in statistical mechanics, entropy is defined as $S = k_B \log \Omega$, where k_B is the boltzman constant, and Ω is something called the multiplicity. The multiplicity counts the number of different configurations of microstates that give rise to the same macroscopic state or description. Quite literally, it is a measure of how little one knows of the microscopic state from the knowledge of the macroscopic. Or, in less information-theoretic words, how chaotic the microscopic state can be without you knowing. Thus, to tie entropy to its familiar "chaotic" or information-theoretic interpretations, we require microstates. Well, the problem with black holes, is that we don't know of any good microstates to describe the matter within them. But with all the matter seemingly at the singularity, we either need to find a way to quantize matter at a singular point or quantize the spacetime around it in such a way that it "leaks out." Well the former is essentially a lost cause: quantum mechanically, they're all too confined to give any meaningful description via our typical quantization techniques (singularities are just bad). And general relativity doesn't speak about microscopic states at all, but if there is a way to quantize spacetime, we might be able to speak about the quantum spectra inside a black hole in a meaningful way. This is a, if not *the*, major theoretical push in high-energy physics today: finding a quantization of spacetime that produces the right entropy for black holes from its microscopic description and reduces to General Relativity in its classical limit.